Conformal anomaly and surface energy for Potts and Ashkin-Teller quantum chains

1987 J. Phys. A: Math. Gen. 20 5677

(http://iopscience.iop.org/0305-4470/20/16/040)

J. Phys. A: Math. Gen. 20 (1987) 5677-5693. Printed in the UK

Conformal anomaly and surface energy for Potts and Ashkin–Teller quantum chains

C J Hamer†‡, G R W Quispel† and M T Batchelor§||

† Department of Theoretical Physics, Research School of Physical Sciences, Australian National University, Canberra, ACT 2601, Australia
§ Department of Mathematics, The Faculties, Australian National University, Canberra, ACT 2601, Australia

Received 8 May 1987

Abstract. Exact equivalences between the critical quantum Potts and Ashkin–Teller chains and a modified *XXZ* Heisenberg chain have recently been derived by Alcaraz *et al.* The leading finite-size corrections to the ground-state energies of these chains are derived using the methods of de Vega, Woynarovich and Eckle. Exact results are then obtained for the conformal anomaly of each model, and for the surface energy in the case of free boundaries.

1. Introduction

There has been great interest recently in calculating the leading finite-size corrections to eigenvalues of the transfer matrix or quantum Hamiltonian for various lattice spin models in two dimensions. Cardy (1984, 1986a) and others (Blöte *et al* 1986, Affleck 1986) have shown that these finite-size corrections are directly related to the critical indices and conformal anomaly of these models by the hypothesis of conformal invariance. If these asymptotic corrections can be derived analytically for the known exactly soluble models, one can either check the validity of conformal invariance from the known exponents, or else, assuming the hypothesis holds, one can obtain exact expressions for various critical exponents.

A method was given previously by de Vega and Woynarovich (1985) for calculating the leading finite-size corrections for any model which is soluble by the Bethe ansatz. They derive integral equations for the finite-size corrections, similar to those which give the bulk ground-state energy. To solve these equations, they use a saddle-point approximation which is valid only when the mass gap is non-zero, i.e. the system is non-critical. It was shown by Hamer (1985) and Avdeev and Dörfel (1986) that this restriction is unnecessary, and that the equations could also be used to give the corrections to the ground-state energy in the critical region. The methods used by these latter authors were somewhat crude, however, and were not capable of giving correction-to-scaling terms (Privman and Fisher 1983) beyond the leading order, or the finite-size behaviour of the excitation spectrum. More recently, Woynarovich and Eckle (1987) have introduced more powerful methods, involving the use of the Euler–Maclaurin formula and a Wiener–Hopf integration, which overcome these difficulties.

‡ Present address: School of Physics, University of New South Wales, Kensington, NSW 2033, Australia.
|| Present address: Instituut-Lorentz voor Theoretische Natuurkunde, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands.

0305-4470/87/165677+17\$02.50

Here we apply these methods to study the ground-state energy of the critical quantum Potts and Ashkin–Teller chains. Both these systems can be exactly related to a modified *XXZ* Heisenberg chain (Hamer 1981, Alcaraz *et al* 1987a, b, c); and Alcaraz *et al* have recently carried out extensive numerical studies on them, calculating the eigenvalue spectrum for chains of up to 512 sites, and extracting accurate values for the conformal anomaly and critical exponents. A very complete picture of their structure has thus been obtained, in agreement with the predictions of conformal invariance. It is our object to confirm some of these results analytically.

Consider then the modified *XXZ* Heisenberg Hamiltonian (Alcaraz *et al* 1987a, b, c)

$$H=-\frac{1}{2}\left(\sum_{j=1}^{N'}(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\Delta\sigma_j^z\sigma_{j+1}^z)+p\sigma_1^z+p'\sigma_N^z\right) \tag{1.1}$$

where N is the number of sites, σ_i^x, σ_i^y, σ_i^z are Pauli matrices, $\Delta=-\cos\gamma$ and $\gamma\in[0,\pi)$. There are three main cases of interest:

(A) $p=p'=0$, $N'=N$, boundary conditions

$$\sigma_{N+1}^x\pm i\sigma_{N+1}^y=e^{i\Phi}(\sigma_1^x\pm i\sigma_1^y)\qquad \sigma_{N+1}^z=\sigma_1^z. \tag{1.2}$$

The eigenvalues of the critical q-state Potts Hamiltonian with periodic boundary conditions on a lattice of M sites can be exactly related (Alcaraz *et al* 1987a, b) to those of chain A with $N=2M$ sites, where $\cos\gamma=\frac{1}{2}\sqrt{q}$ and $\Phi=2\gamma$.

(B) $p=-p'=i\sin\gamma$, $N'=N-1$, free boundaries. The eigenvalues of the critical q-state Potts chain on M sites with free boundaries are related (Alcaraz *et al* 1987a, b, c) to those of the chain B, where again $\cos\gamma=\frac{1}{2}\sqrt{q}$, and $N=2M$.

(C) $p=p'=0$, $N'=N-1$, free boundaries. The eigenvalues of the critical Ashkin–Teller chain on M sites with free boundaries are related (Alcaraz *et al* 1987a, b, c) to those of chain C with $2M$ sites, where the Ashkin–Teller coupling $\lambda=\cos\gamma$.

In § 2 we set out in some detail the treatment of the simple *XXZ* Hamiltonian with periodic boundary conditions. This system has already been dealt with by Woynarovich and Eckle (1987), but it is a useful standard of comparison for the other cases, and serves to keep the paper self-contained. In § 3 the modified Hamiltonian (1.1) is briefly treated as a variation on the theme of § 2; and in § 4 our results for the conformal anomaly and surface energy of the quantum Potts and Ashkin–Teller models are summarised.

2. The *XXZ* model

Consider first the simple *XXZ* model, i.e. equation (1.1) with $N'=N$, the 'surface fields' $p=p'=0$, and periodic boundary conditions. The number of sites N will be assumed even throughout. The Bethe ansatz for this system was discussed in detail by Yang and Yang (1966); we shall use the notation of de Vega and Woynarovich (1985) and Hamer (1985). Questions of mathematical rigour will be neglected, since the system has already been treated by Woynarovich and Eckle (1987).

The total number m of down-spins in the chain is conserved, and the ground state lies in the sector $m=N/2$. The Bethe ansatz for the eigenstates involves a momentum p_j for each down-spin, and the periodic boundary conditions are satisfied if

$$Np_j=2\pi I_j-\sum_{l=1}^{m}\theta(p_j,p_l) \tag{2.1}$$

where

$$\theta(p, q) = 2\tan^{-1}\left(\frac{\Delta\sin[(p-q)/2]}{\cos[(p+q)/2] - \Delta\cos[(p-q)/2]}\right) \tag{2.2}$$

and the I_j are integers or half-odd integers, given by

$$I_1, I_2, \ldots, I_m = -\left(\frac{m-1}{2}\right), -\left(\frac{m-1}{2}\right) + 1, \ldots, +\left(\frac{m-1}{2}\right). \tag{2.3}$$

The energy is

$$E = -\tfrac{1}{2}N\Delta + 2\sum_{j=1}^{m} (\Delta - \cos p_j). \tag{2.4}$$

In the critical region a convenient change of variables is

$$p = 2\tan^{-1}[\cot(\tfrac{1}{2}\gamma)\tanh\lambda] \equiv \varnothing(\lambda, \tfrac{1}{2}\gamma) \qquad (-\infty < \lambda < \infty) \tag{2.5}$$

then (2.1) becomes

$$N\varnothing(\lambda_j, \tfrac{1}{2}\gamma) = 2\pi I_j + \sum_{i=1}^{m} \varnothing(\lambda_j - \lambda_i, \gamma) \tag{2.6}$$

and the energy is

$$E = \tfrac{1}{2}N\cos\gamma - \sin\gamma \sum_{j=1}^{m} \varnothing'(\lambda_j, \tfrac{1}{2}\gamma) \tag{2.7}$$

where the prime denotes differentiation with respect to the λ variable. Note that

$$\varnothing'(\lambda, \tfrac{1}{2}\gamma) = \frac{2\sin\gamma}{\cosh 2\lambda - \cos\gamma}. \tag{2.8}$$

De Vega and Woynarovich (1985) then define the function

$$Z_N(\lambda) = \frac{1}{2\pi}\left(\varnothing(\lambda, \tfrac{1}{2}\gamma) - \frac{1}{N}\sum_{j=1}^{m} \varnothing(\lambda - \lambda_j, \gamma)\right) \tag{2.9}$$

so that

$$Z_N(\lambda_i) = I_i/N \tag{2.10}$$

i.e. the roots are uniformly spaced in the variable Z; the derivative is denoted

$$\sigma_N(\lambda) = \mathrm{d}Z_N/\mathrm{d}\lambda. \tag{2.11}$$

Integrating over all λ one finds

$$\int_{-\infty}^{\infty} \sigma_N(\lambda)\,\mathrm{d}\lambda = \tfrac{1}{2}. \tag{2.12}$$

When N goes to infinity, the roots λ_i tend to a continuous distribution with density $N\sigma_N(\lambda)$ and the sums reduce to integrals. The asymptotic root density satisfies

$$\sigma_\infty(\lambda) = \frac{1}{2\pi}\varnothing'(\lambda, \tfrac{1}{2}\gamma) - \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{2\pi}\sigma_\infty(\mu)\varnothing'(\lambda - \mu, \gamma) \tag{2.13}$$

and the energy per site is

$$e_\infty = \lim_{N\to\infty}(E/N) = \tfrac{1}{2}\cos\gamma - \sin\gamma\int_{-\infty}^{\infty} \mathrm{d}\lambda\,\sigma_\infty(\lambda)\varnothing'(\lambda, \tfrac{1}{2}\gamma). \tag{2.14}$$

The integral equation (2.13) can be solved by a Fourier transform to give

$$\sigma_\infty(\lambda)=[2\gamma\cosh(\pi\lambda/\gamma)]^{-1} \tag{2.15}$$

and hence one obtains the ground-state energy per site (Yang and Yang 1966)

$$e_\infty=\tfrac{1}{2}\cos\gamma-\sin^2\gamma\int_{-\infty}^{\infty}\frac{d\lambda}{\cosh\pi\lambda}\,\frac{1}{[\cosh(2\gamma\lambda)-\cos\gamma]}. \tag{2.16}$$

For finite N, de Vega and Woynarovich (1985) show that one can recast the problem in terms of a similar set of integral equations. After some manipulations, one finds that the difference between the finite and the infinite system is

$$\sigma_N(\lambda)-\sigma_\infty(\lambda)=-\int_{-\infty}^{\infty}\frac{d\mu}{\pi}\,p(\lambda-\mu)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(\mu-\lambda_i)-\sigma_N(\mu)\right) \tag{2.17}$$

for the root density, and

$$e_N-e_\infty=-2\pi\sin\gamma\int_{-\infty}^{\infty}d\lambda\,\sigma_\infty(\lambda)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(\lambda-\lambda_i)-\sigma_N(\lambda)\right) \tag{2.18}$$

for the energy per site, where the kernel $p(\lambda)$ is defined by

$$p(\lambda)=\tfrac{1}{2}\int_{-\infty}^{\infty}d\omega\,e^{i\lambda\omega}\frac{\sinh[(\pi-2\gamma)\omega/2]}{\{\sinh[(\pi-2\gamma)\omega/2]+\sinh(\pi\omega/2)\}} \tag{2.19}$$

and the λ_i are the roots of the finite system. It was shown by Hamer (1985) that the simple approximation $\sigma_N(\lambda)\approx\sigma_\infty(\lambda)$ in (2.18) is sufficient to give the leading-order behaviour of the energy difference, and hence the conformal anomaly for this model.

A more systematic approach has been formulated by Woynarovich and Eckle (1987). Let

$$S_N(\lambda)\equiv\frac{1}{N}\sum_{i=1}^{m}\delta(\lambda-\lambda_i)-\sigma_N(\lambda) \tag{2.20}$$

then one can use the Euler-Maclaurin formula to show that

$$\begin{aligned}\int_{-\infty}^{\infty}g(\lambda)S(\lambda)\,d\lambda\approx&-\left(\int_{-\infty}^{-\Lambda}d\lambda+\int_{\Lambda}^{\infty}d\lambda\right)g(\lambda)\sigma_N(\lambda)\\&+\frac{1}{2N}(g(\Lambda)+g(-\Lambda))+\frac{1}{12N^2\sigma_N(\Lambda)}(g'(\Lambda)-g'(-\Lambda))\end{aligned} \tag{2.21}$$

for an arbitrary function $g(\lambda)$ analytic in $[-\Lambda,\Lambda]$, where Λ is the root of largest magnitude, determined by

$$Z_N(\Lambda)=\frac{1}{4}-\frac{1}{2N} \tag{2.22}$$

or

$$\int_{\Lambda}^{\infty}\sigma_N(\lambda)d\lambda=\frac{1}{2N} \tag{2.23}$$

and the error in (2.21) is $O(N^{-4}d^4g(\xi)/dz^4)$, for some $-\Lambda\leqslant\xi\leqslant\Lambda$. We have used the fact that $\sigma_N(\lambda)$ is symmetric in λ.

Applying this to the energy per site, one finds

$$e_N - e_\infty \approx 4\pi \sin\gamma\left(\int_\Lambda^\infty \mathrm{d}\lambda\, \sigma_\infty(\lambda)\sigma_N(\lambda) - \frac{\sigma_\infty(\Lambda)}{2N} - \frac{\sigma'_\infty(\Lambda)}{12N^2\sigma_N(\Lambda)}\right) \tag{2.24}$$

while for the root density

$$\begin{aligned}\sigma_N(\lambda) - \sigma_\infty(\lambda) = &\int_\Lambda^\infty \mathrm{d}\mu\, \sigma_N(\mu)\frac{p(\lambda-\mu)}{\pi} - \frac{1}{2N\pi}p(\lambda-\Lambda) + \frac{1}{12\pi N^2\sigma_N(\Lambda)}p'(\lambda-\Lambda)\\ &+\left(\int_{-\infty}^{-\Lambda} \mathrm{d}\mu\, \sigma_N(\mu)\frac{p(\lambda-\mu)}{\pi} - \frac{1}{2N\pi}p(\lambda+\Lambda) - \frac{1}{12\pi N^2\sigma_N(\Lambda)}p'(\lambda+\Lambda)\right).\end{aligned} \tag{2.25}$$

Now we are interested in the asymptotic behaviour of the root density for $\lambda \geqslant \Lambda$. In this region the terms inside the large brackets in (2.25) can be treated as small perturbations, and will be dropped henceforth. The remaining equation has a Wiener–Hopf form, similar to (but not identical with) the equation treated by Yang and Yang (1966).

2.1. *The Wiener–Hopf equation*

Define

$$k(\lambda) = p(\lambda)/\pi \tag{2.26}$$

$$f(\lambda) = \sigma_\infty(\lambda+\Lambda) \tag{2.27}$$

and

$$\chi(\lambda) = \sigma_N(\lambda+\Lambda) \tag{2.28}$$

then setting $t = \lambda - \Lambda$ we obtain from equation (2.25)

$$\chi(t) - \int_0^\infty k(t-s)\chi(s)\,\mathrm{d}s \approx f(t) - \frac{1}{2N}k(t) + \frac{1}{12N^2\sigma_N(\Lambda)}k'(t) \tag{2.29}$$

which is the standard form of the Wiener–Hopf equation treated by (for instance) Krein (1962) or Morse and Feshbach (1953). Define Fourier transform pairs $\chi \leftrightarrow X$, $k \leftrightarrow K$, $f \leftrightarrow F$, e.g.,

$$X(\omega) = \int_{-\infty}^\infty e^{i\omega t}\chi(t)\,\mathrm{d}t \tag{2.30}$$

$$\chi(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\omega t}X(\omega)\,\mathrm{d}\omega. \tag{2.31}$$

Then following the standard treatments we first study the kernel of the equation. We have from (2.19)

$$1 - K(\omega) = \frac{\sinh(\pi\omega/2)}{2\sinh[(\pi-\gamma)\omega/2]\cosh(\gamma\omega/2)}. \tag{2.32}$$

Using the decomposition

$$\pi \operatorname{cosec} \pi z = \Gamma(z)\Gamma(1-z) \tag{2.33}$$

to 'factorise' (2.32):

$$1-K(\omega)=\frac{p_+(\omega)p_-(\omega)}{q_+(\omega)q_-(\omega)} \tag{2.34}$$

where $p_\pm(\omega)$, $q_\pm(\omega)$ are entire functions with zeros in the interior of the lower and upper halves $\pi_\mp$ of the complex ω plane, respectively:

$$p_+(\omega)=\frac{1}{\Gamma(1-\mathrm{i}\omega/2)}=p_-(-\omega) \tag{2.35}$$

$$q_+(\omega)=[2(\pi-\gamma)]^{1/2}\left[\Gamma\left(\tfrac{1}{2}-\frac{\mathrm{i}\gamma\omega}{2\pi}\right)\Gamma\left(1-\frac{\mathrm{i}(\pi-\gamma)\omega}{2\pi}\right)\right]^{-1}=q_-(-\omega) \tag{2.36}$$

then we can assert at once that

$$[1-K(\omega)]^{-1}=G_+(\omega)G_-(\omega) \tag{2.37}$$

where

$$G_+(\omega)=[2(\pi-\gamma)]^{1/2}\Gamma(1-\tfrac{1}{2}\mathrm{i}\omega)\mathrm{e}^{\psi(\omega)}\left[\Gamma\left(\tfrac{1}{2}-\frac{\mathrm{i}\gamma\omega}{2\pi}\right)\Gamma\left(1-\frac{\mathrm{i}(\pi-\gamma)\omega}{2\pi}\right)\right]^{-1}=G_-(-\omega). \tag{2.38}$$

Here $G_\pm(\omega)$ are holomorphic and continuous in the half-planes $\pi_\pm$ respectively, while $\psi(\omega)$ is an entire, odd function of ω.

Next determine $\psi(\omega)$ by imposing the condition that $G_\pm(\omega)$ are continuous and equal to 1 at $|\omega|\to\infty$ (in $\pi_\pm$, respectively). Stirling's formula may then be used to show that

$$\psi(\omega)=\frac{\mathrm{i}\omega}{2}\left[\ln\left(\frac{\pi}{\pi-\gamma}\right)-\frac{\gamma}{\pi}\ln\left(\frac{\gamma}{\pi-\gamma}\right)\right]. \tag{2.39}$$

Now we return to the Fourier transformed version of our original equation (2.29). The functions $X(\omega)$ and $F(\omega)$ may be split into $\pm$ components (holomorphic and continuous in $\pi_\pm$, respectively), by e.g.,

$$X(\omega)=X_+(\omega)+X_-(\omega) \tag{2.40}$$

$$X_\pm(\omega)=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\chi_\pm(t)\,\mathrm{d}t \tag{2.41}$$

$$\chi_\pm(t)=\begin{cases}\chi(t) & \text{for } t\gtrless 0\\ 0 & \text{for } t\lessgtr 0.\end{cases} \tag{2.42}$$

Then (2.29) gives rise to the equation

$$X_-(\omega)+(1-K(\omega))(X_+(\omega)-C(\omega))=F_+(\omega)+F_-(\omega)-C(\omega) \tag{2.43}$$

where $C(\omega)$ is an entire function

$$C(\omega)=\frac{1}{2N}+\frac{\mathrm{i}\omega}{12N^2\sigma_N(\Lambda)}. \tag{2.44}$$

Now use (2.37) to obtain

$$\frac{X_+(\omega)-C(\omega)}{G_+(\omega)}+G_-(\omega)[X_-(\omega)+C(\omega)-F_-(\omega)]-G_-(\omega)F_+(\omega)=0. \tag{2.45}$$

Our aim is to separate out terms which are analytic in the two half-planes $\pi_\pm$. The final term is not in the required form and must be further split:

$$G_-(\omega)F_+(\omega)=Q_+(\omega)+Q_-(\omega). \tag{2.46}$$

Then one can write

$$(X_+(\omega)-C(\omega))/G_+(\omega)-Q_+(\omega)=Q_-(\omega)-G_-(\omega)[X_-(\omega)+C(\omega)-F_-(\omega)]\equiv P(\omega). \tag{2.47}$$

The left-hand side of this equation is analytic in π_+, the right-hand side in π_-. There is a common strip of regularity including the junction of the half-planes $\pi_\pm$, so the right-hand side is the analytic continuation of the left-hand side into π_-. The function $P(\omega)$ so defined is therefore *entire*, and can be determined from the asymptotic behaviour of one of its defining expressions. Hence we may solve for $X_+(\omega)$ and $X_-(\omega)$.

As $|\omega|\to\infty$ in π_+, $X_+(\omega)\to 0$ (from the requirement that $\chi_+(t)$ be integrable at the origin), $Q_+(\omega)\to 0$ (see below), and using asymptotic expansions for the Γ functions in (2.38) one finds

$$G_+(\omega)\underset{|\omega|\to\infty}{\sim}1+\frac{g_1}{\omega}+\frac{g_2}{\omega^2}+\mathrm{O}(\omega^{-3}) \tag{2.48}$$

where

$$g_1=\frac{\mathrm{i}}{12}\left(2+\frac{\pi}{\gamma}-\frac{2\pi}{\pi-\gamma}\right)\qquad g_2=\tfrac{1}{2}g_1^2. \tag{2.49}$$

Hence one obtains

$$P(\omega)=\frac{\mathrm{i}g_1}{12N^2\sigma_N(\Lambda)}-\frac{1}{2N}-\frac{\mathrm{i}\omega}{12N^2\sigma_N(\Lambda)}. \tag{2.50}$$

From (2.15) and (2.27) we have

$$F(\omega)=\frac{\mathrm{e}^{-\mathrm{i}\omega\Lambda}}{2\cosh(\frac{1}{2}\gamma\omega)}. \tag{2.51}$$

Keeping only the leading pole term in π_-, we therefore have

$$F_+(\omega)\approx\frac{\mathrm{e}^{-\mathrm{i}\omega\Lambda}}{\pi-\mathrm{i}\gamma\omega}. \tag{2.52}$$

Therefore

$$G_-(\omega)F_+(\omega)=(\pi-\mathrm{i}\gamma\omega)^{-1}\{[G_-(\omega)\,\mathrm{e}^{-\mathrm{i}\omega\Lambda}-G_-(-\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda/\gamma}]$$
$$+G_-(-\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda/\gamma}\} \tag{2.53}$$

where the pole term has been added and subtracted again to show that

$$Q_+(\omega)=\frac{G_-(-\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda/\gamma}}{(\pi-\mathrm{i}\gamma\omega)}=\frac{G_+(\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda/\gamma}}{(\pi-\mathrm{i}\gamma\omega)}. \tag{2.54}$$

Finally, then, we obtain a solution from (2.47):

$$X_+(\omega)=\int_0^\infty \mathrm{e}^{\mathrm{i}\omega t}\,\chi(t)\,\mathrm{d}t=C(\omega)+G_+(\omega)[P(\omega)+Q_+(\omega)] \tag{2.55}$$

where the functions on the right-hand side are known.

2.2. *Results*

The solution (2.55) may now be used to determine the finite-size corrections. First, recall the constraint (2.23)

$$\frac{1}{2N}=\int_{\Lambda}^{\infty}\sigma_N(\lambda)\,\mathrm{d}\lambda=X_+(0)=\frac{1}{2N}+G_+(0)[P(0)+Q_+(0)]. \tag{2.56}$$

Therefore

$$Q_+(0)=-P(0)$$

that is

$$\frac{G_+(\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda/\gamma}}{\pi}=\frac{1}{2N}-\frac{\mathrm{i}g_1}{12N^2\sigma_N(\Lambda)}. \tag{2.57}$$

Next, from the definition (2.28),

$$\sigma_N(\Lambda)=\chi(0)=2\chi_+(0) \tag{2.58}$$

where

$$\chi_+(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\,X_+(\omega)=\frac{g_1^2}{48N^2\sigma_N(\Lambda)}+\frac{\mathrm{i}g_1}{4N}+\frac{G_+(\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda/\gamma}}{2\gamma} \tag{2.59}$$

obtained by contour integration. The factor 2 in (2.58) appears because $\chi_+(t)$ drops discontinuously to zero at $t=0$, so the Fourier transform (2.59) gives only half the full $\chi(0)$.

Substituting (2.57) into (2.59), one finds the result

$$\sigma_N(\Lambda)=\frac{1}{4N}\left\{\frac{\pi}{\gamma}+\mathrm{i}g_1+\left[\left(\frac{\pi}{\gamma}\right)^2+\frac{2}{3}\frac{\mathrm{i}\pi g_1}{\gamma}-\frac{1}{3}g_1^2\right]^{1/2}\right\} \tag{2.60}$$

showing an explicit $1/N$ dependence. Finally, approximating (2.15)

$$\sigma_\infty(\Lambda)\approx\gamma^{-1}\,\mathrm{e}^{-\pi\Lambda/\gamma} \tag{2.61}$$

and using (2.57) and (2.60), the equation (2.24) gives rise after a little algebra to:

$$e_N-e_\infty=-\pi^2\sin\gamma/6\gamma N^2 \tag{2.62}$$

as derived previously by Hamer (1985).

There are corrections to this result, which arise from the four approximations made in the derivation. Firstly, the approximations at (2.25) and (2.61) are essentially equivalent: the next order terms give rise to corrections $O(\mathrm{e}^{-3\pi\Lambda/\gamma}/N)=O(N^{-4})$ in (2.62), using (2.57) and (2.60). Secondly, there is the error made in the approximation (2.21), which is also $O(N^{-4})$. Finally, there are the bracketed terms in (2.25) which were dropped. These introduce corrections† $O(p(2\Lambda)/N)$ into the determination of $\sigma_N(\Lambda)$ and $\mathrm{e}^{-\pi\Lambda/\gamma}$. Now from (2.19),

$$p(\lambda)=\tfrac{1}{4}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}\lambda\omega}\{1-\coth[(\pi-\gamma)\omega/2]\tanh(\tfrac{1}{2}\gamma\omega)\}. \tag{2.63}$$

† This argument is actually oversimplified. More careful consideration shows that the perturbative correction $\delta\sigma_N(\lambda)$ obeys a higher-order Wiener–Hopf equation, in which the terms discussed here correspond only to the inhomogeneous piece. However, we have checked that this naive argument does give the order of magnitude correctly for the cases discussed in this paper.

This function has been analysed by Yang and Yang (1966). Completing the contour in the upper half-plane, the leading poles at $\omega = \mathrm{i}\pi/\gamma$ and $\omega = 2\mathrm{i}\pi/(\pi-\gamma)$ imply

$$p(\lambda) \underset{\lambda\to\infty}{\sim} c_1 \mathrm{e}^{-\pi\lambda/\gamma} + c_2 \mathrm{e}^{-2\pi\lambda/(\pi-\gamma)} \tag{2.64}$$

and so

$$p(2\Lambda)/N \underset{N\to\infty}{\sim} c_1' N^{-3} + c_2' N^{-1-4\gamma/(\pi-\gamma)}. \tag{2.65}$$

Hence finally we find

$$e_N - e_\infty = -(\pi^2 \sin\gamma/6\gamma N^2)[1 + \mathrm{O}(N^{-2}) + \mathrm{O}(N^{-4\gamma/(\pi-\gamma)})]. \tag{2.66}$$

These correction terms were first obtained (and are discussed in more detail) by Woynarovich and Eckle (1987).

The limiting case $\gamma\to 0$ requires separate consideration. It may be treated by taking the limit $\gamma\to 0$ in the formulae above, after first rescaling variables as follows: $\lambda\to\gamma\lambda$, $\omega\to\omega/\gamma$, $\sigma\to\sigma/\gamma$, $p\to p/\gamma$. One finds that to leading order

$$e_N - e_\infty = -\pi/6N^2 \qquad (\gamma = 0) \tag{2.67}$$

as expected. The rescaled kernel is

$$p(\lambda) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{\mathrm{i}\lambda\omega}}{1+\mathrm{e}^{|\omega|}} \underset{\lambda\to\infty}{\sim} \frac{1}{4\lambda^2}. \tag{2.68}$$

Hence one finds the correction-to-scaling terms are logarithmic in this case,

$$e_N - e_\infty = -(\pi/6N^2)[1 + \mathrm{O}(\ln N)^{-3}] \tag{2.69}$$

as discussed in more detail by Woynarovich and Eckle (1987).

3. The Potts and Ashkin–Teller models

As outlined in the introduction, equivalences can be found (Alcaraz *et al* 1987a, b, c) between the eigenvalues of the critical Potts and Ashkin–Teller chains and those of the modified XXZ Hamiltonian (1.1). Here we shall consider two particular versions.

(A) Same as case A in the introduction;

(B) $N' = N-1$, free boundaries. Cases B and C of the introduction are then special cases of this.

In each case, the finite-size corrections to the ground-state energy per site can be found following a very similar procedure to that of § 2. Here we shall merely indicate the significant differences which occur in each case.

3.1. The Bethe ansatz equations

The boundary conditions imply for case A (Alcaraz *et al* 1987a, b) with $m = N/2$

$$Np_j = 2\pi I_j + \Phi - \sum_{l=1}^{m} \theta(p_j, p_l) \qquad I_j = -\tfrac{1}{2}(m+1) + j \qquad j = 1, \ldots, m \tag{3.1}$$

or

$$N\varnothing(\lambda_j, \tfrac{1}{2}\gamma) = 2\pi I_j + \Phi + \sum_{l=1}^{m} \varnothing(\lambda_j - \lambda_l, \gamma). \qquad \text{(A)} \tag{3.2}$$

For case B, one finds (Alcaraz *et al* 1987c) the corresponding equation ($m = N/2$)

$$2N\varnothing(\lambda_j, \tfrac{1}{2}\gamma) = 2\pi I_j - \varnothing(\gamma_j, \Gamma) - \varnothing(\lambda_j, \Gamma') + \sum_{\substack{l=1\\(\neq j)}}^{m} [\varnothing(\lambda_j - \lambda_l, \gamma) + \varnothing(\lambda_j + \lambda_l, \gamma)]$$

$$I_j = j \qquad j = 1, \ldots, m \qquad \text{(B)} \tag{3.3}$$

where

$$e^{2i\Gamma} = \frac{p - \Delta - e^{i\gamma}}{(p-\Delta)\, e^{i\gamma} - 1} \qquad e^{2i\Gamma'} = \frac{p' - \Delta - e^{i\gamma}}{(p'-\Delta)\, e^{i\gamma} - 1}. \tag{3.4}$$

The energy is

$$E = -\tfrac{1}{2}N\Delta + 2\sum_{j=1}^{m} (\Delta - \cos p_j) \qquad \text{(A)} \tag{3.5}$$

$$E = -(N-1)\Delta/2 - \tfrac{1}{2}(p + p') + 2\sum_{j=1}^{m} (\Delta - \cos p_j). \qquad \text{(B)} \tag{3.6}$$

Changing variables as in (2.9), the function $Z_N(\lambda)$ can once again be defined such that $Z_N(\lambda_i) = I_i/N$, as follows:

$$Z_N(\lambda) = \frac{1}{2\pi}\left(\varnothing(\lambda, \tfrac{1}{2}\gamma) - \frac{\Phi}{N} - \frac{1}{N}\sum_{j=1}^{m} \varnothing(\lambda - \lambda_j, \gamma)\right) \qquad \text{(A)} \tag{3.7}$$

$$Z_N(\lambda) = \frac{1}{\pi}\left(\varnothing(\lambda, \tfrac{1}{2}\gamma) + \frac{1}{2N}[\varnothing(\lambda, \Gamma) + \varnothing(\lambda, \Gamma') + \varnothing(2\lambda, \gamma)]\right.$$
$$\left. - \frac{1}{2N}\sum_{j=1}^{m} [\varnothing(\lambda - \lambda_j, \gamma) + \varnothing(\lambda + \lambda_j, \gamma)]\right). \qquad \text{(B)} \tag{3.8}$$

In fact, for case B the root density is symmetric about $\lambda = 0$, and so we can rewrite (3.8)

$$Z_N(\lambda) = \frac{1}{\pi}\left(\varnothing(\lambda, \tfrac{1}{2}\gamma) + \frac{1}{2N}[\varnothing(\lambda, \Gamma) + \varnothing(\lambda, \Gamma') + \varnothing(2\lambda, \gamma) + \varnothing(\lambda, \gamma)]\right.$$
$$\left. - \frac{1}{2N}\sum_{j=-m}^{m} \varnothing(\lambda - \lambda_j, \gamma)\right). \qquad \text{(B)} \tag{3.9}$$

The sum rules corresponding to (2.12) are

$$\int_{-\infty}^{\infty} d\lambda\, \sigma_N(\lambda) = \begin{cases} \frac{1}{2} & \text{(A)} \quad (3.10) \\ 1 + N^{-1}[3 - 2(\gamma + \Gamma + \Gamma')/\pi]. & \text{(B)} \quad (3.11) \end{cases}$$

3.2. *The thermodynamic limit*

The resulting integral equations valid when $N \to \infty$ are

$$\sigma_\infty(\lambda) = \frac{1}{2\pi}\varnothing'(\lambda, \tfrac{1}{2}\gamma) - \int_{-\infty}^{\infty} \frac{d\mu}{2\pi}\sigma_\infty(\mu)\varnothing'(\lambda - \mu, \gamma) \qquad \text{(A)} \tag{3.12}$$

$$\sigma_\infty(\lambda) = \frac{1}{\pi}\varnothing'(\lambda, \tfrac{1}{2}\gamma) - \int_{-\infty}^{\infty} \frac{d\mu}{2\pi}\sigma_\infty(\mu)\varnothing'(\lambda - \mu, \gamma) \qquad \text{(B)} \tag{3.13}$$

and

$$e_\infty = \tfrac{1}{2}\cos\gamma - \sin\gamma \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \sigma_\infty(\lambda)\varnothing'(\lambda, \tfrac{1}{2}\gamma) \qquad \text{(A)} \tag{3.14}$$

$$e_\infty = \tfrac{1}{2}\cos\gamma - \tfrac{1}{2}\sin\gamma \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \sigma_\infty(\lambda)\varnothing'(\lambda, \tfrac{1}{2}\gamma). \qquad \text{(B)} \tag{3.15}$$

The solutions are

$$\sigma_\infty(\lambda) = \begin{cases} [2\gamma\cosh(\pi\lambda/\gamma)]^{-1} & \text{(A)} \quad (3.16) \\ [\gamma\cosh(\pi\lambda/\gamma)]^{-1} & \text{(B)} \quad (3.17) \end{cases}$$

$$e_\infty = \tfrac{1}{2}\cos\gamma - \sin^2\gamma \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{\cosh(\pi\lambda)} \frac{1}{\cosh(2\gamma\lambda) - \cos\gamma}. \qquad \text{(A, B)} \tag{3.18}$$

The asymptotic root density is twice as large in the case with free boundaries, but the energy per site is the same, as of course it must be.

3.3. *Finite-size corrections*

The integral equations giving the finite-size corrections are:

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\pi} p(\lambda-\mu)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(\mu-\lambda_i) - \sigma_N(\mu)\right) \qquad \text{(A)} \tag{3.19}$$

and

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\pi} p(\lambda-\mu)\left(\frac{1}{N}\sum_{i=-m}^{m}\delta(\mu-\lambda_i) - \sigma_N(\mu)\right) + \frac{1}{\pi N}[p_1(\lambda,\Gamma) + p_1(\lambda,\Gamma') + p(\lambda) + p_2(\lambda)] \qquad \text{(B)} \tag{3.20}$$

where

$$p_1(\lambda,\Gamma) = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\lambda} \frac{\sinh[(\pi-2\Gamma)\omega/2]}{\sinh[(\pi-2\gamma)\omega/2] + \sinh(\pi\omega/2)} \tag{3.21}$$

$$p_2(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\lambda} \frac{\sinh[(\pi-2\gamma)\omega/4]\cosh(\pi\omega/4)}{\sinh[(\pi-2\gamma)\omega/2] + \sinh(\pi\omega/2)} \tag{3.22}$$

while

$$e_N - e_\infty = -2\pi\sin\gamma \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \sigma_\infty(\lambda)\left(\frac{1}{N}\sum_{i=1}^{m}\delta(\lambda-\lambda_i) - \sigma_N(\lambda)\right) \qquad \text{(A)} \tag{3.23}$$

$$e_N - e_\infty = -\tfrac{1}{2}\pi\sin\gamma \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \sigma_\infty(\lambda)\left(\frac{1}{N}\sum_{i=-m}^{m}\delta(\lambda-\lambda_i) - \sigma_N(\lambda)\right) - \frac{1}{2N}\left(\cos\gamma + p + p' - \pi\sin\gamma\,\sigma_\infty(0) + \frac{\sin\gamma}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \varnothing'(\lambda, \tfrac{1}{2}\gamma) \times [p_1(\lambda,\Gamma) + p_1(\lambda,\Gamma') + p_2(\lambda)]\right). \qquad \text{(B)} \tag{3.24}$$

The Euler-Maclaurin formula can be applied as before to give:

$$e_N - e_\infty \approx 2\pi \sin\gamma\left[\left(\left(\int_{\Lambda_+}^{\infty} d\lambda + \int_{-\infty}^{-\Lambda_-}\right)\sigma_\infty(\lambda)\sigma_N(\lambda) - (1/2N)(\sigma_\infty(\Lambda_+) + \sigma_\infty(-\Lambda_-)\right)\right.$$
$$\left. - \frac{1}{12N^2}\left(\frac{\sigma'_\infty(\Lambda_+)}{\sigma_N(\Lambda_+)} - \frac{\sigma'_\infty(-\Lambda_-)}{\sigma_N(-\Lambda_-)}\right)\right] \qquad \text{(A)} \qquad (3.25)$$

$$e_N - e_\infty \approx \pi \sin\gamma\left(\int_\Lambda^\infty d\lambda\, \sigma_\infty(\lambda)\sigma_N(\lambda) - \frac{\sigma_\infty(\Lambda)}{2N} - \frac{\sigma'_\infty(\Lambda)}{12N^2\sigma_N(\Lambda)}\right)$$
$$-\frac{1}{2N}\left(\cos\gamma + p + p' - \pi\sin\gamma\,\sigma_\infty(0)\right.$$
$$\left. + \frac{\sin\gamma}{\pi}\int_{-\infty}^{\infty} d\lambda\, \varnothing'(\lambda, \tfrac{1}{2}\gamma)[p_1(\lambda,\Gamma) + p_1(\lambda,\Gamma') + p_2(\lambda)]\right) \qquad \text{(B)} \qquad (3.26)$$

and

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) \approx \left(\int_{\Lambda_+}^{\infty} d\mu + \int_{-\infty}^{-\Lambda_-} d\mu\right)\frac{p(\lambda-\mu)}{\pi}\sigma_N(\mu) - \frac{1}{2N\pi}[p(\lambda-\Lambda_+) + p(\lambda+\Lambda_-)]$$
$$-\frac{1}{12N^2}\left(\frac{p'(\lambda+\Lambda_-)}{\sigma_N(-\Lambda_-)} - \frac{p'(\lambda-\Lambda_+)}{\sigma_N(\Lambda_+)}\right) \qquad \text{(A)} \qquad (3.27)$$

$$\sigma_N(\lambda) - \sigma_\infty(\lambda) \approx \left(\int_{\Lambda}^{\infty} d\mu + \int_{-\infty}^{-\Lambda} d\mu\right)\frac{p(\lambda-\mu)}{\pi}\sigma_N(\mu) - \frac{1}{2N\pi}[p(\lambda-\Lambda) + p(\lambda+\Lambda)]$$
$$-\frac{1}{12N^2\pi\sigma_N(\Lambda)}[p'(\lambda+\Lambda) - p'(\lambda-\Lambda)]$$
$$+\frac{1}{\pi N}[p_1(\lambda,\Gamma) + p_1(\lambda,\Gamma') + p_2(\lambda) + p(\lambda)] \qquad \text{(B)} \qquad (3.28)$$

where the cutoffs are given by

$$\int_{\Lambda_+}^{\infty} \sigma_N(\lambda)\, d\lambda = \frac{1}{2N}\left(1 - \frac{\Phi}{\pi}\right) \qquad \int_{-\infty}^{-\Lambda_-} \sigma_N(\lambda)\, d\lambda = \frac{1}{2N}\left(1 + \frac{\Phi}{\pi}\right) \qquad \text{(A)} \qquad (3.29)$$

$$\int_\Lambda^\infty \sigma_N(\lambda)\, d\lambda = \frac{1}{2N}\left(3 - \frac{2}{\pi}(\gamma + \Gamma + \Gamma')\right). \qquad \text{(B)} \qquad (3.30)$$

Note that $\sigma_N(\lambda)$ and the cutoffs $\Lambda_\pm$ are not symmetric in case A, as they are in case B. In order for the right-hand side of (3.30) to be positive, we henceforth impose the condition $\gamma + \Gamma + \Gamma' \leqslant 3\pi/2$.

3.4. Wiener-Hopf equations

The Wiener-Hopf equations are the same in each case, and in particular the kernel functions $K(\omega)$, $G_+(\omega)$ are identical to those in § 2. The only differences lie in the inhomogenous term:

$$Q_+(\omega) \approx \frac{G_+(i\pi/\gamma)\, e^{-\pi\Lambda_\pm/\gamma}}{(\pi - i\gamma\omega)} \qquad \text{(A)} \qquad (3.31)$$

$$Q_+(\omega) \approx \frac{2G_+(i\pi/\gamma)\, e^{-\pi\Lambda/\gamma}}{(\pi - i\gamma\omega)}. \qquad \text{(B)} \qquad (3.32)$$

3.5. Results

Applying the constraints (3.29) and (3.30) we obtain

$$\frac{G_+(\mathrm{i}\pi/\gamma)\,\mathrm{e}^{-\pi\Lambda_\pm/\gamma}}{\pi}=\frac{1}{2N}-\frac{\mathrm{i}g_1}{12N^2\sigma_N(\pm\Delta_\pm)}\mp\frac{\Phi}{2\pi NG_+(0)} \qquad \text{(A)} \tag{3.33}$$

$$\frac{2G_+(\mathrm{i}\pi/\gamma)\mathrm{e}^{-\pi\Lambda/\gamma}}{\pi}=\frac{1}{2N}-\frac{\mathrm{i}g_1}{12N^2\sigma_N(\Lambda)}+\frac{\alpha}{N}$$

$$\alpha=\frac{1}{G_+(0)}\left[1-\left(\frac{\gamma+\Gamma+\Gamma'}{\pi}\right)\right]. \qquad \text{(B)} \tag{3.34}$$

Hence for the root density at the cutoff:

$$\sigma_N(\pm\Lambda_\pm)=\frac{1}{4N}\left\{\frac{\pi}{\gamma}+\mathrm{i}g_1\mp\frac{\Phi}{\gamma G_+(0)}\right.$$
$$\left.+\left[\left(\frac{\pi}{\gamma}\right)^2+\frac{2}{3}\frac{\mathrm{i}\pi g_1}{\gamma}-\frac{1}{3}g_1^2+\frac{\Phi^2}{\gamma^2G_+(0)^2}\mp\frac{2\Phi}{\gamma G_+(0)}\left(\frac{\pi}{\gamma}+\mathrm{i}g_1\right)\right]^{1/2}\right\} \qquad \text{(A)} \tag{3.35}$$

$$\sigma_N(\Lambda)=\frac{1}{4N}\left\{\frac{\pi}{\gamma}+\mathrm{i}g_1+\frac{2\pi\alpha}{\gamma}\right.$$
$$\left.+\left[\left(\frac{\pi}{\gamma}\right)^2+\frac{2}{3}\frac{\mathrm{i}\pi g_1}{\gamma}-\frac{1}{3}g_1^2+\frac{4\pi^2\alpha^2}{\gamma}+\frac{4\pi\alpha}{\gamma}\left(\frac{\pi}{\gamma}+\mathrm{i}g_1\right)\right]^{1/2}\right\}. \qquad \text{(B)} \tag{3.36}$$

Then finally for the energy per site

$$e_N-e_\infty\approx-\frac{\pi^2\sin\gamma}{6\gamma N^2}\left(1-\frac{3\Phi^2}{\pi^2G_+(0)^2}\right) \qquad \text{(A)} \tag{3.37}$$

$$e_N-e_\infty\approx-\frac{\pi^2\sin\gamma}{24\gamma N^2}(1-12\alpha^2)-\frac{1}{2N}\left(\cos\gamma+p+p'-\pi\sin\gamma\,\sigma_\infty(0)\right.$$
$$\left.+\frac{\sin\gamma}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\;\varnothing'(\lambda,\tfrac{1}{2}\gamma)[p_1(\lambda,\Gamma)+p_1(\lambda,\Gamma')+p_2(\lambda)]\right). \qquad \text{(B)} \tag{3.38}$$

The corrections to these results can be derived along the same lines as in § 2. The leading terms in case B come from the 'surface terms' in (3.28),

$$\mathrm{O}\{[p(\Lambda)+p_2(\Lambda)+p_1(\Lambda,\Gamma)+p_1(\Lambda,\Gamma')]/N\}.$$

Hence one finds that the leading corrections to (3.37) and (3.38) are

$$\frac{1}{N^2}[\mathrm{O}(N^{-2})+\mathrm{O}(N^{-4\gamma/(\pi-\gamma)})] \qquad \text{(A)} \tag{3.39}$$

$$\frac{1}{N^2}[\mathrm{O}(N^{-1})+\mathrm{O}(N^{-2\gamma/(\pi-\gamma)})]. \qquad \text{(B)} \tag{3.40}$$

In the limiting case $\gamma\to 0$, after rescaling one finds:

$$e_N-e_\infty\approx-\frac{\pi^2}{6N^2}\left(1-\frac{3\Phi^2}{\pi^2G_+(0)^2}\right) \qquad \text{(A)} \tag{3.41}$$

$$e_N - e_\infty \approx -\frac{\pi^2}{24N^2} - \frac{1}{2N}\left(1+p+p'-\pi\sigma_\infty(0) + \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda\, \varnothing'(\lambda, \tfrac{1}{2}\gamma)[p_1(\lambda,\tilde{\Gamma})+p_1(\lambda,\tilde{\Gamma}')+p_2(\lambda)]\right) \qquad \text{(B)} \qquad (3.42)$$

where the rescaled functions of interest are:

$$\sigma_\infty(\lambda) = \frac{1}{\cosh(\pi\lambda)} \qquad (3.43)$$

$$p(\lambda) = \int_0^\infty d\omega \frac{\cos(\lambda\omega)}{1+e^\omega} \underset{\lambda\to\infty}{\sim} \frac{1}{4\lambda^2} \qquad (3.44)$$

$$p_2(\lambda) = \frac{1}{2}\int_0^\infty d\omega \frac{\cos(\lambda\omega)}{\cosh(\omega/2)} \underset{\lambda\to\infty}{\sim} O(\lambda^{-4}) \qquad (3.45)$$

and

$$p_1(\lambda,\tilde{\Gamma}) = \frac{1}{2}\int_0^\infty d\omega \frac{\cos(\lambda\omega)}{\cosh(\omega/2)} e^{(1-2\tilde{\Gamma})\omega/2} \underset{\lambda\to\infty}{\sim} \frac{(2\tilde{\Gamma}-1)}{4\lambda^2} \qquad (3.46)$$

with

$$\tilde{\Gamma} = \lim_{\gamma\to 0}\left(\frac{\Gamma}{\gamma}\right) \qquad \tilde{\Gamma}' = \lim_{\gamma\to 0}\left(\frac{\Gamma'}{\gamma}\right).$$

Hence the leading correction to $e_N - e_\infty$ is $O(p(\Lambda)/N^2) = O((\ln N)^{-2}/N^2)$ in case B. For case A we find the coefficient at this order vanishes as for the simple XXZ model (Woynarovich and Eckle 1987), so that the leading correction is $O((\ln N)^{-3}/N^2)$.

4. Conclusions

The finite-size scaling form of the ground state energy per site is predicted by conformal invariance (Blöte *et al* 1986, Affleck 1986) to be:

$$e_N = e_\infty - \frac{\pi\zeta c}{6N^2} + o(N^{-2}) \qquad \text{(periodic boundary conditions)} \qquad (4.1)$$

$$e_N = e_\infty + \frac{f_\infty}{N} - \frac{\pi\zeta c}{24N^2} + o(N^{-2}) \qquad \text{(free boundary conditions).} \qquad (4.2)$$

Here e_∞ and f_∞ are, respectively, the bulk limits of the ground state energy per site and surface energy, c is the conformal anomaly which governs the critical exponents of the system, and ζ is a scale factor which is independent of the boundary conditions and is known (Hamer 1985) to be

$$\zeta = (\pi \sin\gamma)/\gamma \qquad (4.3)$$

for the XXZ Hamiltonian (1.1). The value of e_∞ is the same for all the models discussed in this paper, and is given by equation (2.16).

To translate the results of § 3 for the cases A, B, and C of the introduction, we need to make the following replacements:

(A) Potts model with periodic boundaries:

set $\quad \Phi = 2\gamma \quad \cos\gamma = \frac{1}{2}\sqrt{q} \quad$ in case A of § 3.

(B) Potts model with free boundaries:

$$\text{set} \qquad p = -p' = \mathrm{i} \sin\gamma \qquad \Gamma' = \pi - \Gamma$$

$$\alpha = -\frac{\gamma}{\pi G_+(0)} = \frac{-\gamma}{[2\pi(\pi-\gamma)]^{1/2}} \qquad \text{in case B of § 3.}$$

(C) Ashkin–Teller model with free boundaries:

$$\text{set} \qquad p = p' = 0 \qquad \Gamma = \Gamma' = \tfrac{1}{2}(\pi - \gamma) \qquad \alpha = 0$$

in case B of § 3.

Thus one arrives at the following conclusions. For the q-state Potts model with either periodic or free boundary conditions, the conformal anomaly has been shown to be

$$c(q) = 1 - \frac{12\gamma^2}{\pi^2(G_+(0))^2} = 1 - \frac{6\gamma^2}{\pi(\pi-\gamma)}$$

where

$$\cos\gamma = \frac{\sqrt{q}}{2} \qquad 0 \leqslant \gamma \leqslant \pi/2. \tag{4.4}$$

This agrees with the values predicted from conformal invariance (Blöte *et al* 1986, Affleck 1986, Dotsenko 1984), with the numerical results of Alcaraz *et al* (1987a, b, c), and with other analytic derivations by Kadanoff and Nienhuis (quoted in Friedan *et al* (1984)) who studied the four-point correlation function and by de Vega and Karowski (1987) who derived the finite-size scaling behaviour of the free energy in the equivalent six-vertex model with 'seam'.

For the Ashkin–Teller model with free boundaries, the conformal anomaly has been shown to be

$$c(\lambda) \equiv 1 \qquad \text{all } \lambda \tag{4.5}$$

which is the value predicted from conformal invariance because of the non-universal behaviour of its critical exponents. It also agrees with the numerical calculations of von Gehlen and Rittenberg (1987) and Alcaraz *et al* (1987a, b, c).

The surface energy f_∞ is a non-universal quantity. For the Potts model with free boundaries our result from (3.38) reduces to

$$f_\infty = \frac{\pi\sin\gamma}{2\gamma} - \frac{\cos\gamma}{2} - \frac{\sin\gamma}{4}\int_{-\infty}^{\infty} \mathrm{d}k[1 - \coth(\tfrac{1}{4}\pi k)\tanh(\tfrac{1}{2}\gamma k)] \tag{4.6}$$

while for the Ashkin–Teller model with free boundaries one obtains

$$f_\infty = \frac{\pi\sin\gamma}{2\gamma} - \frac{\cos\gamma}{2} - \frac{\sin\gamma}{4}\int_{-\infty}^{\infty} \mathrm{d}k[1 - \tanh(\tfrac{1}{4}\pi k)\tanh(\tfrac{1}{2}\gamma k)]. \tag{4.7}$$

Upon evaluating these expressions, the results are in precise agreement† with the numerical estimates of Alcaraz *et al* (1987c).

† All our results quoted are actually for the equivalent XXZ models. We have not translated them back to the original Potts or Ashkin–Teller models.

The leading correction-to-scaling terms have also been estimated in § 3. For the corrections to $e_N - e_\infty$ we find:

(A) $N^{-2}[\mathrm{O}(N^{-2}) + \mathrm{O}(N^{-4\gamma/(\pi-\gamma)})]$ for the Potts model with periodic boundaries ($\gamma > 0$), just as for the XXZ model (Woynarovich and Eckle 1987),

(B) $N^{-2}[\mathrm{O}(N^{-1}) + \mathrm{O}(N^{-4\gamma/(\pi-\gamma)})]$ for the Potts model with free boundaries† ($\gamma > 0$),

(C) $N^{-2}[\mathrm{O}(N^{-1}) + \mathrm{O}(N^{-2\gamma/(\pi-\gamma)})]$ for the Ashkin–Teller model with free boundaries ($\gamma > 0$).

In the limit $\gamma \to 0$ the corresponding results are:

(A) $N^{-2}[\mathrm{O}((\ln N)^{-3})]$ as for the XXZ model, and as predicted by Cardy (1986b) from conformal invariance;

(B, C) $N^{-2}[\mathrm{O}((\ln N)^{-2})]$.

Note that case A, $\gamma \to 0$, corresponds to the four-state Potts model. We have confirmed that the coefficient of the leading correction term in this case is given by:

$$e_N - e_\infty \underset{N\to\infty}{\sim} -\frac{\pi\zeta}{6N^2}\left(1 + \frac{0.3433}{(\ln N)^3}\right)$$

as was found by Woynarovich and Eckle (1987). This coefficient differs from the value $\frac{3}{4}$ predicted by Cardy (1986b) on the basis of renormalisation group arguments.

Finally, Alcaraz *et al* (1987a, b) have found that the difference in the ground-state energy for $\Phi = \pi$ and $\Phi = 0$ in case A gives the mass gap amplitude corresponding to the Ashkin–Teller polarisation exponent. Our results show that for this case

$$e_N(\Phi = \pi) - e_N(\Phi = 0) = \frac{\pi^3 \sin\gamma}{4\gamma N^2(\pi - \gamma)} + \mathrm{O}(N^{-3}) \underset{N\to\infty}{\sim} \frac{\pi \sin\gamma}{\gamma N^2} 2\pi x \tag{4.8}$$

where x is the scaling dimension of the polarisation operator; and therefore

$$x = \frac{\pi}{8(\pi - \gamma)}. \tag{4.9}$$

This provides an analytic confirmation of the results of Alcaraz *et al* (1987a, b).

It is possible to generalise the above slightly. Equation (3.3) is real provided

$$\begin{aligned} &\text{(i)} \quad (p - \Delta)(p' - \Delta) = 1 \quad \text{or} \\ &\text{(ii)} \quad p^* = p' \quad \text{or} \\ &\text{(iii)} \quad \mathrm{Im}\, p = \mathrm{Im}\, p'. \end{aligned} \tag{4.10}$$

In each of these three cases the conformal anomaly is given by

$$c = 1 - \frac{6}{\pi(\pi - \gamma)}\left[\gamma - \tan^{-1}\left(\frac{1}{1 - R}\tan\gamma\right)\right]^2 \tag{4.11}$$

where‡

$$R = \frac{p + p'}{\Delta - \Delta(p - \Delta)(p' - \Delta)}. \tag{4.12}$$

† The coefficient of the term $\mathrm{O}(N^{-2\gamma/(\pi-\gamma)})$ quoted in § 3 vanishes for this case.

‡ It is interesting to observe that this corresponds to the conformal anomaly for a Coulomb-like system with a charge $-2\alpha = \gamma - \tan^{-1}[(1 - R)^{-1}\tan\gamma](\pi^2 - \gamma\pi)^{-1/2}$ at infinity (Dotsenko and Fateev 1984).

This would indicate that the Potts model is attained as a singular limit in parameter space. We hope to report on this more extensively in the future.

Using the methods of Woynarovich and Eckle (1987), we have thus been able to derive the finite-size scaling behaviour of the ground-state energy of the critical Potts and Ashkin–Teller chains, using the equivalences between these and the modified *XXZ* chains found by Alcaraz *et al* (1987a, b, c). These methods can also be used to explore the spectrum of excited states, as shown by Woynarovich (1987) for the *XXZ* model. We hope to carry out this exercise for the Potts and Ashkin–Teller chains in future work.

Acknowledgment

One of the authors (MTB) gratefully acknowledges support from a Commonwealth Postgraduate Research Award.

References

Affleck I 1986 *Phys. Rev. Lett.* **56** 746
Alcaraz F C, Barber M N and Batchelor M T 1987a *Phys. Rev. Lett.* **58** 771
—— 1987b to be published
Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987c *J. Phys. A: Math. Gen.* to be published
Avdeev L V and Dörfel B-D 1986 *J. Phys. A: Math. Gen.* **19** L13
Blöte H W J, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L385
—— 1986a *Nucl. Phys.* B **270** [FS16] 186
—— 1986b *J. Phys. A: Math. Gen.* **19** L1093
de Vega H J and Karowski M 1987 *Preprint* CERN TH 4631/87
de Vega H J and Woynarovich F 1985 *Nucl. Phys.* B **251** 439
Dotsenko V S 1984 *Nucl. Phys.* B **235** [FS11] 54
Dotsenko V S and Fateev V A 1984 *Nucl. Phys.* B **240** [FS12] 312
Friedan D, Qiu Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
Hamer C J 1981 *J. Phys. A: Math. Gen.* **14** 2981
—— 1985 *J. Phys. A: Math. Gen.* **18** L1133
—— 1986 *J. Phys. A: Math. Gen.* **19** 3335
Krein M G 1962 *Am. Math. Soc. Translations, Series 2* **22** 163
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw-Hill) p 978
Privman V and Fisher M E 1983 *J. Phys. A: Math. Gen.* **16** L295
von Gehlen G and Rittenberg V 1987 *J. Phys. A: Math. Gen.* **20** 227
Woynarovich F 1987 *Phys. Rev. Lett.* **59** 259
Woynarovich F and Eckle H-P 1987 *J. Phys. A: Math. Gen.* **20** L97
Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321, 327